INFORMATION AND CONTROL 10, 254–291 (1967)

Detection Theory and Quantum Mechanics

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Statistical signal detection is formulated quantum-mechanically in terms of choosing one of two density operators as the better description of the state of an ideal receiver after exposure to a field in which a signal may or may not be present. The optimum decision procedure is expressed as a projection operator on the state-space of the receiver. Examples involving the single-mode detection of coherent and incoherent signals are given. Threshold detection as an approximation to optimum detection for weak signals is also defined quantum-mechanically, and threshold detectors for coherent and incoherent fields occupying many modes of a receiver cavity are worked out. The noise in all cases is taken to be thermal radiation described by the Planck law.

LIST OF SYMBOLS

a	Annihilation operator
a^+	Adjoint of a = creation operator
$\mathbf{A}$, $\mathbf{B}$, $\mathbf{M}$	Row vectors of mode amplitudes α_k, β_k, μ_k
b, b'	Constants in threshold operator
c	Velocity of light
$\bar{C}$	Average cost of operation
C_{ij}	Cost of choosing hypothesis H_i when hypothesis H_j is true
d^2	Power signal-to-noise ratio
$d^n\mathbf{x}$	$dx_1\, dx_2 \cdots dx_n$ = volume element in space of $\mathbf{x}$
$d\Omega$	Element of solid angle
$dE(\theta)$	Infinitesimal projection operator for estimating parameter θ
$dR(\theta)$	Region in space of outcomes x leading to parameter estimate in $(\theta, \theta + d\theta)$
D^2	Equivalent signal-to-noise ratio

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e	$2.71828 \cdots$
$\mathbf{e}_\delta$	Polarization vector ($\delta = 1, 2$)
$\mathbf{E}$	Expectation value
E	Signal energy
$\boldsymbol{\varepsilon}$	Electric field operator
$f(u)$	Generating function of Laguerre distribution
$f(\mathbf{x})$	Decision statistic
$F_k(\alpha^*)$	Glauber representation of eigenstate of $\rho_1 - \lambda\rho_0$
g, g_i	Decision statistics
$g(u)$	Generating function of cumulative Laguerre distribution
G	Sum of g_i's
$\hbar$	Planck's constant $= h/2\pi$
H	Hamiltonian operator
H_i	Hypothesis
i	$\sqrt{-1}$, or (as subscript) integral-valued index
$\mathbf{I}$	Identity matrix
$I_0(x)$	Modified Bessel function of order 0
j, k, m, n	Integral-valued indices (as subscripts)
$\mathbf{k}, \mathbf{m}$	Quadruple mode indices
K	Boltzmann's constant
$\mathbf{K}$	Matrix in Eqs. (4.14), (5.3)
L_i	Length of side of cavity
$L_k(x)$	Laguerre polynomial of order k
M	Number of exposures to source of signal
n	Number operator a^+a; the number of photons in a mode
n_0	The decision level on n
N	Mean number of photons
p	Momentum operator
$p_0(\mathbf{x})$	Probability density function of measurements $\mathbf{x}$ under hypothesis H_0
$p_1(\mathbf{x}; \theta)$	Probability density function of measurements $\mathbf{x}$ under hypothesis H_1
p_k	Probabilities in Laguerre distribution (Appendix B)
$\mathbf{P}$	Matrix [exp $(u\mathbf{Q}) - \mathbf{I}$]
P_{ik}	Probability that system is in state $\mid k\rangle$ under hypothesis H_i
$P_0(\beta)$	Density function for thermal radiation
q	Co-ordinate operator

$q_{\mathbf{km}}$	Coefficients in quadratic threshold operator
q_k	Cumulative Laguerre distribution (Appendix B)
$\mathbf{Q}$	Matrix of coefficients $q_{\mathbf{km}}$
$Q(\alpha^*, \beta)$	Glauber representation of detection operator Π_θ
$Q(\alpha, \beta)$	Marcum's Q-function (Section 2, Appendix B)
Q'	Quadratic part of threshold operator
Q_0	False-alarm probability
Q_d	Probability of detection
$\mathbf{r}$	Position vector
$R_i(\alpha^*, \beta)$	Glauber representation of density operator ρ_i ($i = 0, 1$)
t	Time
T	Duration of interval when aperture is open
$\mathfrak{T}$	Absolute temperature
u	Variable in moment-generating function; Lagrange multiplier
$\mathbf{u}_{\mathbf{m}}(\mathbf{r})$	Mode eigenfunction
v	Lagrange multiplier
v_i	Parameter in distribution of thermal radiation; $v_0 = e^{-w}$
V	Volume of cavity
w	$\hbar\omega/K\mathfrak{T}$ = parameter in density operator for thermal radiation
$\mathbf{x}$	Set of measurements of receiver input
X	Dynamical variable to be measured and its quantum-mechanical operator
x_m	Eigenvalues of X
$z(\theta)$	Prior probability density function of parameter θ
$Z_{\mathbf{mk}}$	Coupling constant
$\mathbf{1}$	Identity operator
α, β, γ	Complex mode amplitudes, or right-eigenvalues of operator a
δ	Polarization index (Eq. (4.3))
δ_{mn}	Kronecker delta
ζ, ζ_i	Prior probabilities
η_k	Eigenvalues of operator $\rho_1 - \lambda\rho_0$
θ	Parameter of density operator (Section 1), signal strength (Section 3)
$\boldsymbol{\theta}$	Unit vector specifying direction of radiation
λ	Lagrangian multiplier and threshold on likelihood ratio
$\mu, \mu_{\mathbf{k}}$	Complex mode amplitudes of signal field

ν	Number of modes in cavity
π	$3.14159 \cdots$
π_k	Probability that hypothesis H_1 is chosen when the outcome of a measurement of X is x_k
$\Pi, \Pi_i, \Pi_\theta, \Pi_0$	Detection operators
Π_{mn}	Matrix elements of Π
ρ_0, ρ_1, ρ_i	Density operators
$\sigma_x, \sigma_y, \sigma_z, \sigma_\psi$	Spin matrices
$\boldsymbol{\varphi}$	Mode correlation matrix
$\Phi(\omega; \boldsymbol{\theta})$	Spectral distribution of radiation
ψ	Auxiliary angle (Section 1), signal phase (Sections 2, 4)
ω	Angular frequency = ordinary frequency times 2π

INTRODUCTION

The development of communication and radar systems using laser beams has stimulated interest in the efficient detection of signals of optical frequencies and in the properties of channels utilizing such signals. (Gordon, 1962, 1964; Jelsma and Bolgiano, 1965; Takahasi, 1965; Lebedev and Levitin, 1966). The reliability of detectors of optical signals is limited not only by the random noise accompanying the signals and generated in the detectors, but also by the quantum nature of the signals themselves, which introduces an additional stochastic element to the detection process. The fundamental limitations on the detectability of signals in ordinary radar and communication systems have been delineated by the statistical theory of signal detection (Peterson *et al.* 1954; Middleton and Van Meter, 1955a; Middleton, 1960, 1965), and it is appropriate to ask what the theory can say about detecting signals of optical frequencies.

Before the limitations on signal detectability can be analyzed, it is necessary to adopt some model for an ideal receiver. This model should involve a minimum of assumptions about the way information is to be extracted from the incident electromagnetic radiation. The usual instruments for detecting optical signals admit the radiation through an aperture into a system that processes it in some way. The radiation may be focused on a sensitive cell or a photographic plate, it may be transmitted to photomultipliers or photon counters, or, when it is coherent, it may be heterodyned with a locally generated coherent beam through some material nonlinearity. Whatever is done to the radiation happens in a limited region of space behind the aperture of the optical system.

A natural idealization of a receiver of optical signals, therefore, is a box or a cavity, initially empty, that is exposed to the source of the signals by opening an aperture during the time when a signal, if transmitted, is expected to arrive. At the end of this time the aperture is closed, and an observer measures the field inside the cavity as extensively as he can in order to extract all the information relevant to a decision whether a signal was present or not in the external electromagnetic field (Takahasi, 1965).

At the low frequencies with which communication theory has in the past been concerned, classical physics adequately describes the electromagnetic fields of signal and noise. It permits the electromagnetic field in the cavity to be measured in as great a detail as necessary, and signal-detection theory has been able to presume that this field is in principle completely known to the observer. At optical frequencies, however, it is necessary to describe the field quantum-mechanically, and quantum mechanics places certain limitations on the precision with which the field can be measured both temporally and spatially. Detection theory must now prescribe not only how the measurements of the field shall be processed, but also what measurements shall be made.

The electromagnetic field in the cavity of our ideal receiver will not be in a pure quantum-mechanical state, but in a statistical mixture of states. Such mixtures are described by density operators (von Neumann, 1932; Fano, 1957). If there was no signal present in the external field during the time when the aperture was open, the field in the cavity after the aperture was closed resulted only from the stochastic background radiation and is represented by a density operator ρ_0. If a signal arrived while the aperture was open, the field is represented by some other density operator ρ_1. The task of the observer is to choose one or the other of these density operators as the more consistent with as much as he can measure of the cavity field.

The first section of this paper reformulates detection theory in terms of such a choice between density operators, preserving the standard goals of minimizing the average cost of operation or, for a fixed false-alarm probability, maximizing the probability of detection. The optimum procedure for deciding between two hypotheses about the cavity field is characterized as the measurement of a certain projection operator. When the two density operators commute, the usual likelihood-ratio strategy appears. Examples are presented in Section 2.

In important cases, however, the density operators between which a choice must be made do not commute. The mathematical problem of finding the optimum projection operator in those cases is a formidable one, and as a means of avoiding it we investigate the common threshold approximation that the signal is weak. In Section 3 a quantum-mechanical form of the optimum threshold detector is defined as the measurement of the operator yielding the greatest equivalent signal-to-noise ratio. An equation for this operator in terms of the density operators ρ_0 and ρ_1 is given. Section 4 derives the threshold detector for a coherent signal of random phase, such as an ideal laser pulse, received in the presence of thermal background radiation. In Section 5 the reception of an incoherent, noise-like signal is treated on the same basis. In both cases the maximum equivalent signal-to-noise ratios reduce at low frequencies to the familiar forms.

1. THE DETECTION OPERATOR

(i) *The Decision Problem in Quantum Mechanics.* Detection in quantum mechanics involves deciding which of two density operators ρ_0 or ρ_1 describes a system. These operators are presumed to be known functions of the operators for the dynamical variables of the system. The hypothesis that ρ_0 applies we denote by H_0, the hypothesis that ρ_1 applies by H_1. The decision between them is to be based on the outcome of a measurement of some dynamical variable X, which is represented by an operator that will also be called X.

This operator X may stand for a set or n-tuple of commuting and hence simultaneously measurable operators. It possesses a set of eigenkets $|x_k\rangle$ corresponding to the eigenvalues x_k :

$$X|x_k\rangle = x_k|x_k\rangle, \tag{1.1}$$

and this set will be assumed to be complete, so that any state of the system can be expressed as a linear combination of the eigenkets $|x_k\rangle$. We shall assume that the eigenvalues form a discrete set and are all distinct. That the eigenvalues, which may also be n-tuples, are distinct means that all degeneracies have been resolved by introducing additional commuting operators, as when the degenerate energy eigenstates of the hydrogen atom are resolved into simultaneous eigenstates of the angular momentum and its component along an arbitrary axis. If the eigenvalues form a continuous set, the usual modifications, involving the replacement of sums by integrals, can be made (Dirac, 1947).

The outcome of a measurement of X is one of the eigenvalues of X, say the mth, x_m ; and the system is left in the associated eigenstate $|x_m\rangle$. No further measurements will be of any help in deciding which of the density operators originally described the system. The observer needs a strategy for choosing one or the other density operator on the basis of the outcome x_m .

A randomized strategy assigns a probability π_m to each outcome x_m and directs the observer to choose hypothesis H_1 with probability π_m—and H_0 with probability $1 - \pi_m$—when that outcome occurs, perhaps by tossing a properly biased coin. In effect he measures not X, in whose value he is not really interested, but a dynamical variable corresponding to the operator

$$\Pi = \sum_k |x_k\rangle\pi_k\langle x_k|, \tag{1.2}$$

which we shall call the "detection operator." The states $|x_k\rangle$ are eigenstates of Π with eigenvalues π_k , and the measurement of Π gives the observer the probability with which he should choose hypothesis H_1 .

The detection operator Π is a Hermitian operator, and as shown in Appendix A, its matrix elements Π_{mn} in any representation are equal to or less in absolute value than 1, and its diagonal elements Π_{mm} are non-negative real numbers:

$$|\,\Pi_{mn}\,| \leqq 1, \qquad 0 \leqq \Pi_{mm} \leqq 1. \tag{1.3}$$

There is an infinity of such detection operators Π, and the problem now is to find the best one.

(ii) *The Detection Criteria.* According to detection theory, receivers should be designed to meet one of two principal criteria, the Bayes criterion or the Neyman-Pearson criterion. The former directs us to minimize the average cost of operation, the latter to maximize the probability of detecting the signal while maintaining a fixed false-alarm probability (Middleton, 1960, Chapt. 19; Helstrom, 1960, Chapt. 3).

The false-alarm probability Q_0 is the probability of choosing hypothesis H_1 when hypothesis H_0 is true. The probability under hypothesis H_0 that a measurement of X or Π will leave the system in the mth state $|x_m\rangle$ is equal to $\langle x_m|\,\rho_0\,|x_m\rangle$, and the total probability that hypothesis H_1 will be selected is

$$Q_0 = \sum_m \pi_m\langle x_m|\,\rho_0\,|x_m\rangle = \mathrm{Tr}\,(\rho_0\Pi), \tag{1.4}$$

where "Tr" stands for the trace of the operator written after it. The probability of detection Q_d is the probability of choosing H_1 when H_1 is true and is similarly given by

$$Q_d = \mathrm{Tr}\,(\rho_1 \Pi). \tag{1.5}$$

Let the prior probabilities of hypotheses H_0 and H_1 be ζ and $(1 - \zeta)$, respectively, and let C_{ij} be the cost of choosing hypothesis H_i when H_j is true $(i, j = 0, 1)$. Then the average cost of each decision is

$$\begin{aligned}\bar{C} &= \zeta[C_{00}(1 - Q_0) + C_{10}Q_0] + (1 - \zeta)[C_{01}(1 - Q_d) + C_{11}Q_d] \\ &= \zeta C_{00} + (1 - \zeta)C_{01} - (1 - \zeta)(C_{01} - C_{11})(Q_d - \lambda Q_0),\end{aligned} \tag{1.6}$$

where

$$\lambda = \frac{\zeta}{1 - \zeta}\left(\frac{C_{10} - C_{00}}{C_{01} - C_{11}}\right). \tag{1.7}$$

If we remember that $C_{01} > C_{11}$, $C_{10} > C_{00}$, we see that we must pick as our detection operator one that maximizes the quantity

$$Q_d - \lambda Q_0 = \mathrm{Tr}\,[(\rho_1 - \lambda\rho_0)\Pi]. \tag{1.8}$$

To meet the Neyman-Pearson criterion we must maximize Q_d for a fixed value of Q_0. By introducing the Lagrange multiplier λ, we find that it is again the quantity $Q_d - \lambda Q_0$ that is to be maximized. The resulting detection operator Π will be a function of λ, which must be determined afterward in such a way that the false-alarm probability Q_0 takes on its pre-assigned value.

(iii) *The Optimum Strategy.* We adopt a representation in which the matrix of the operator $\rho_1 - \lambda\rho_0$ is diagonal, and we denote its eigenvalues by η_k and its associated eigenstates by $|\eta_k\rangle$:

$$\rho_1 - \lambda\rho_0 = \sum_k |\eta_k\rangle\eta_k\langle\eta_k|. \tag{1.9}$$

If in that representation the matrix elements of Π are Π_{mn}, we are to maximize the quantity

$$\mathrm{Tr}\,[(\rho_1 - \lambda\rho_0)\Pi] = \sum_m \eta_m \Pi_{mm}.$$

Since the diagonal elements Π_{mm} are positive real numbers between 0 and 1, this quantity is largest if we take $\Pi_{nn} = 1$ for $\eta_n \geqq 0$ and $\Pi_{nn} = 0$ for $\eta_n < 0$. The relation

$$\sum_k |\Pi_{nk}|^2 \leqq 1$$

(Appendix A, Eq. (A.4)) then requires the off-diagonal elements Π_{mn} to vanish in those rows m and columns n for which either $\Pi_{mm} = 1$ or $\Pi_{nn} = 1$. In the rest of the matrix they vanish by virtue of the relation $|\Pi_{mn}| \leqq \Pi_{mm}\Pi_{nn}$ (Eq. (A.6)), for there both $\Pi_{mm} = 0$ and $\Pi_{nn} = 0$. Hence the matrix $\|\Pi_{mn}\|$ is diagonal with eigenvalues 0 and 1.

The optimum detection operator is therefore a projection operator on to the manifold spanned by the eigenkets of the operator $\rho_1 - \lambda\rho_0$ with non-negative eigenvalues:

$$\Pi = \sum_{k:\eta_k \geqq 0} |\eta_k\rangle\langle\eta_k|. \tag{1.10}$$

As in ordinary choices between simple alternatives, the observer can adopt a nonrandomized strategy. He measures the dynamical variable whose operator is $\rho_1 - \lambda\rho_0$, and if the outcome of his measurement is non-negative, he chooses hypothesis H_1, otherwise H_0. The false-alarm and detection probabilities attaining the minimum Bayes cost are given by

$$Q_0 = \sum_{k:\eta_k \geqq 0} \langle\eta_k|\,\rho_0\,|\eta_k\rangle, \qquad Q_d = \sum_{k:\eta_k \geqq 0} \langle\eta_k|\,\rho_1\,|\eta_k\rangle, \tag{1.11}$$

and in order to calculate them it is necessary to be able to find the eigenvalues and eigenstates of $\rho_1 - \lambda\rho_0$.

(iv) *The Choice Between Two Directions of Spin.* The optimum detection operator Π can seldom be easily calculated when the density operators ρ_0 and ρ_1 do not commute. Here is an example whose only value is simplicity and instructiveness. Someone is sending a beam of spin-$\frac{1}{2}$ particles, such as sodium atoms, along the y-axis, preparing it in such a way that each particle has its spin parallel either to the z-axis (hypothesis H_0) or to the x-axis (hypothesis H_1). For each particle the observer is to choose between these hypotheses.

In terms of the Pauli spin matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.12}$$

(Dirac, 1947, p. 149), the two density operators between which one is to decide are

$$\rho_0 = \tfrac{1}{2}(\mathbf{I} + \sigma_z), \qquad \rho_1 = \tfrac{1}{2}(\mathbf{I} + \sigma_x) \tag{1.13}$$

where $\mathbf{I}$ is the 2×2 density matrix. These density operators do not commute. The optimum detection operator Π, which maximizes

$$\mathrm{Tr}\,[(\rho_1 - \lambda\rho_0)\Pi] = \tfrac{1}{2}\,\mathrm{Tr}\,\{[(1-\lambda)\mathbf{I} + \sigma_x - \lambda\sigma_z]\Pi\},$$

can be shown to be

$$\Pi = \tfrac{1}{2}(\mathbf{I} - \sigma_z \cos\psi + \sigma_x \sin\psi), \tag{1.14}$$

where the angle ψ is defined by $\lambda = \cot\psi$ and lies between 0 and $\pi/2$.

The eigenvalues of $\rho_1 - \lambda\rho_0$ are

$$\eta_1 = \tfrac{1}{2}[1 - \cot(\psi/2)] \leqq 0, \tag{1.15}$$

$$\eta_2 = \tfrac{1}{2}[1 + \tan(\psi/2)] > 0,$$

and the eigenstates, in a representation in which, as in Eq. (1.12), σ_z is diagonal, are

$$|\eta_1\rangle = \begin{pmatrix} \cos(\psi/2) \\ -\sin(\psi/2) \end{pmatrix}, \qquad |\eta_2\rangle = \begin{pmatrix} \sin(\psi/2) \\ \cos(\psi/2) \end{pmatrix}. \tag{1.16}$$

We leave the calculations to the reader.

Both the operator $\rho_1 - \lambda\rho_0$ and the detection operator Π involve the spins only through the operator

$$\sigma_\psi = \sigma_z \cos\psi - \sigma_x \sin\psi, \tag{1.17}$$

whose only eigenvalues are $+1$ and -1. This operator can be measured for each particle by passing the beam through an inhomogeneous magnetic field directed at an angle $\psi + \pi/2$ with respect to the x-axis. This field splits the beam into two components. For the particles of one component σ_ψ has the eigenvalue $+1$, and these, one decides, were originally spinning in the z-direction. For the particles of the other component, the eigenvalue of σ_ψ is -1, and to them one assigns hypothesis H_1.

The false-alarm and detection probabilities are

$$Q_0 = \mathrm{Tr}\,(\rho_0\Pi) = \tfrac{1}{2}(1 - \cos\psi) \tag{1.18}$$

$$Q_d = \mathrm{Tr}\,(\rho_1\Pi) = \tfrac{1}{2}(1 + \sin\psi) \tag{1.19}$$

as can be easily calculated from Eqs. (1.13), (1.14) if one remembers the rules for the spin matrices,

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbf{I}, \qquad \sigma_z\sigma_x = i\sigma_y,$$

$$\mathrm{Tr}\,\sigma_x = \mathrm{Tr}\,\sigma_y = \mathrm{Tr}\,\sigma_z = 0, \qquad \mathrm{Tr}\,\mathbf{I} = 2.$$

Thus

$$Q_d = \tfrac{1}{2} + [Q_0(1 - Q_0)]^{1/2}, \qquad 0 \leqq Q_0 \leqq \tfrac{1}{2},\ \tfrac{1}{2} \leqq Q_d \leqq 1. \tag{1.20}$$

A value $Q_0 = 0$ corresponds to $\psi = 0$. The magnetic field is then directed along the z-axis, and whenever the particle is spinning in the z-direction, it goes into the component beam with the eigenvalue of σ_ψ equal to 1, and hypothesis H_0 is correctly chosen. If it is spinning in the x-direction, it will give the eigenvalues $+1$ and -1 with equal probabilities, and hypothesis H_1 is correctly chosen with probability $Q_d = \frac{1}{2}$. For $Q_0 = \frac{1}{2}$, $Q_d = 1$, the field lies in the x-direction. The best orientation of the field is determined, under the Bayes criterion, by the value of $\lambda = \cot \psi$ given by Eq. (1.7). Under the Neyman-Pearson criterion the angle ψ is determined by the pre-assigned false-alarm probability through Eq. (1.18).

For problems of any real interest, unfortunately, it seems to be very difficult to diagonalize the operator $\rho_1 - \lambda\rho_0$ when ρ_0 and ρ_1 do not commute; and unless this can be done, the optimum detection operator cannot be found. We therefore examine in Section 3 an alternative based on the presumption that the signal to be detected is relatively weak.

(v) *Commuting Density Operators.* Matters are much simpler when the density operators ρ_1 and ρ_0 commute, for it is then necessary to measure only one or the other or, if available, a dynamical variable X whose operator commutes with both. Two commuting density operators possess a common set of eigenkets, which we denote by $|k\rangle$, and they can be written as

$$\rho_i = \sum_k |k\rangle P_{ik} \langle k|, \qquad i = 0, 1, \quad (1.21)$$

where P_{ik} is the probability that the system is in the state $|k\rangle$ under hypothesis H_i ($i = 0, 1$). The eigenvalues of $\rho_1 - \lambda\rho_0$ are now

$$\eta_k = P_{1k} - \lambda P_{0k}, \qquad (1.22)$$

and the optimum strategy is to choose hypothesis H_1 if a measurement of ρ_0, ρ_1, or X leaves the system in a state $|k\rangle$ for which

$$P_{1k}/P_{0k} \geqq \lambda. \qquad (1.23)$$

This is the familiar likelihood-ratio strategy of detection theory. If the common eigenstates of the density operators ρ_0 and ρ_1 form a continuous rather than a discrete set, the probabilities P_{0k} and P_{1k} are replaced by probability density functions.

(vi) *Multiple Choices and Parameter Estimation.* The question how best to choose among more than two hypotheses can be formulated in a similar way. Denote the hypotheses by $H_1, H_2, \cdots, H_r$ and define r commuting operators $\Pi_1, \Pi_2, \cdots, \Pi_r$ whose sum is the identity operator,

$$\Pi_1 + \Pi_2 + \cdots + \Pi_r = \mathbf{1}.$$

These operators are simultaneously measured for the system, and the outcomes give the probabilities with which the corresponding hypotheses should be adopted, again by means of a chance device. If ζ_i is the prior probability of hypothesis H_i and C_{ij} is the cost of choosing hypothesis H_i when H_j is true, the average cost is

$$\bar{C} = \sum_{i=1}^{r} \sum_{j=1}^{r} \zeta_j C_{ij} \operatorname{Tr} (\rho_j \Pi_i), \tag{1.24}$$

where ρ_j is the density operator of the system under hypothesis H_j. The operators Π_j, $j = 1, 2, \cdots, r$, must be selected to minimize this average cost. If the density operators ρ_i commute, any one of them, or a sufficient statistic X commuting with all of them, can be measured, and the treatment of the outcome is the same as in ordinary multi-hypothesis decision theory (Middleton and Van Meter, 1955b).

In the quantum-mechanical counterpart of the problem of parameter estimation, the density operator of a system is of a known functional form, but depends on an unknown parameter θ. The statement that the value of θ lies in the interval $(\theta, \theta + d\theta)$ is what von Neumann (1932) calls a "property" ("Eigenschaft") of the system and identifies with a projection operator $dE(\theta)$: $\int dE(\theta) = \mathbf{1}$. If the true values θ_0 of the parameter have a prior probability density function $z(\theta_0)$, and if $C(\theta, \theta_0)$ is the cost of assigning the value θ to the parameter when its true value is θ_0, the Bayes cost of estimating θ by means of a particular "resolution of the identity" $dE(\theta)$ is

$$\bar{C} = \iint z(\theta_0) C(\theta, \theta_0) \operatorname{Tr} [\rho(\theta_0)\, dE(\theta)]\, d\theta_0. \tag{1.25}$$

One must find the set $dE(\theta)$ of infinitesimal projection operators that minimizes this average cost. If X is the best dynamical variable to measure, and if its operator has the continuous eigenvalues x and the eigenkets $|x\rangle$,

$$dE(\theta) = \int_{dR(\theta)} |x\rangle\langle x|\, dx, \tag{1.26}$$

where $dR(\theta)$ is the region in the space of outcomes x leading to an estimate of the parameter in the range $(\theta, \theta + d\theta)$.

2. DETECTION IN A SINGLE MODE

(i) *The Harmonic Oscillator.* Some simple examples to illustrate the ideas of the previous section can be put forth by considering the field in the cavity of our ideal receiver to have a single mode that can be excited by coupling with the external electromagnetic field. In quantum mechanics this mode can be treated as a simple harmonic oscillator of frequency ω and unit mass. The co-ordinate q and the momentum p of the oscillator are expressed in terms of an "annihilation operator" a and a "creation operator," the adjoint a^+ of a:

$$q = (\hbar/2\omega)^{1/2}(a^+ + a), \qquad p = i(\hbar\omega/2)^{1/2}(a^+ - a), \tag{2.1}$$

where $\hbar$ is Planck's constant $h/2\pi$. The operator a and its adjoint a^+ are subject to the commutation relations

$$aa^+ - a^+a = 1. \tag{2.2}$$

The operator $n = a^+a$ is called the "number operator." We denote its eigenstates by $|m\rangle$.

$$n|m\rangle = m|m\rangle. \tag{2.3}$$

It is customary to say that when the oscillator or the mode is in the state $|m\rangle$, it contains m "photons"; and a representation of the state in terms of the eigenstates $|m\rangle$ of the number operator $n = a^+a$ is called the "number representation." Since the Hamiltonian of the oscillator is

$$H = \tfrac{1}{2}(p^2 + \omega^2q^2) = (\hbar\omega/2)(aa^+ + a^+a) = \hbar\omega(n + \tfrac{1}{2}), \tag{2.4}$$

the eigenstates of the number operator n are stationary states of the harmonic oscillator. The effects of the annihilation and creation operators on such an eigenstate $|m\rangle$ are given by the equations

$$a|m\rangle = m^{1/2}|m - 1\rangle, \qquad a^+|m\rangle = (m + 1)^{1/2}|m + 1\rangle. \tag{2.5}$$

Thus the action of a is to decrease the number of photons by 1, and the action of a^+ is to increase it by 1, whence the names of these operators.

R. J. Glauber (1963) has developed a useful calculus for dealing with the states of a harmonic oscillator in terms of the right eigenkets of the operator a, which are denoted by $|\alpha\rangle$:

$$a|\alpha\rangle = \alpha|\alpha\rangle. \tag{2.6}$$

The α's are complex numbers ranging over the entire complex plane, and the set of eigenkets $|\alpha\rangle$ is overcomplete. Nevertheless any state of the oscillator can be represented as a superposition of these eigenkets by virtue of the representation of the identity,

$$\mathbf{1} = \pi^{-1} \int |\alpha\rangle\langle\alpha| \, d^2\alpha, \tag{2.7}$$

in which the integration is taken over the plane of $\alpha = \alpha_x + i\alpha_y$, and $d^2\alpha = d\alpha_x \, d\alpha_y$. These states $|\alpha\rangle$ in the co-ordinate representation are Gaussian wave-packets of minimum uncertainty $\Delta p \Delta q$. When $|\alpha|$ is large, they exhibit the features of a classical oscillating field of frequency ω, and they can be considered as the counterparts of coherent signals in classical electromagnetism. The average energy of the state $|\alpha\rangle$ is

$$\langle\alpha| H |\alpha\rangle = \hbar\omega(|\alpha|^2 + \tfrac{1}{2}). \tag{2.8}$$

We shall suppose that when there is no signal present, the oscillator is in a mixture of states characteristic of thermal radiation. Its density operator ρ_0 can then be written as

$$\begin{aligned} \rho_0 = (1 - e^{-w})e^{-wn} &= \sum_{k=0}^{\infty} P_{0k} \, |k\rangle\langle k| \\ &= (\pi N)^{-1} \int \exp\,(-|\alpha|^2/N) \, |\alpha\rangle\langle\alpha| \, d^2\alpha, \end{aligned} \tag{2.9}$$

where N is the mean number of photons in the mode,

$$N = v_0/(1 - v_0), \qquad v_0 = N/(N + 1) = e^{-w} \tag{2.10}$$

(Louisell, 1964, p. 243; Glauber, 1963, p. 2780). The eigenvalues P_{0k} of the density operator are given by the geometrical distribution

$$P_{0k} = (1 - v_0)v_0^k. \tag{2.11}$$

In thermal equilibrium at absolute temperature $\mathfrak{J}$ the mean number N is determined by the Planck formula,

$$N = (e^w - 1)^{-1}, \qquad w = \hbar\omega/K\mathfrak{J} \tag{2.12}$$

with K the Boltzmann constant.

(ii) *An Incoherent Signal.* By an "incoherent signal" we mean one that superimposes on the distribution of states given by ρ_0 another distribution of the same kind, but with a mean number of photons N_s.

The density operator ρ_1 under hypothesis H_1 has the same form as that in Eq. (2.9), except that N is replaced by $N' = N + N_s$, and v_0 is replaced by $v_1 = N'/(N' + 1)$.[1]

The two density operators ρ_0 and ρ_1 commute and possess the simultaneous eigenstates $|m\rangle$, for both are functions only of the number operator n, which can be considered as a sufficient statistic. A decision about the presence or absence of the signal can be based on the outcome of a measurement of n. If the outcome of this measurement exceeds a decision level n_0, hypothesis H_1 is chosen. Under the Bayes criterion,

$$n_0 = \frac{\ln [\lambda(1 - v_0)/(1 - v_1)]}{\ln (v_1/v_0)}. \tag{2.13}$$

The false-alarm and detection probabilities are

$$Q_0 = v_0^{n_0'}, \qquad Q_d = v_1^{n_0'}, \tag{2.14}$$

where n_0' is the least integer greater than n_0.

(iii) *A Coherent Signal of Known Phase.* Let us suppose that when a coherent signal of known phase impinges on our cavity, the single mode with which we are dealing is excited into a coherent state $|\mu\rangle$, which as we mentioned before is a right eigenstate of the annihilation operator a corresponding to the complex eigenvalue μ. If there is also background radiation of the type described by the density operator ρ_0 of Eq. (2.9), the combined fields of the signal and the background are distributed among the possible states of the mode oscillator in accordance with the density operator

$$\begin{aligned}\rho_1 &= (1 - e^{-w})e^{-w(a^+ - \mu^*)(a-\mu)} \\ &= (\pi N)^{-1} \int \exp(-|\alpha - \mu|^2/N)\, |\alpha\rangle\langle\alpha|\, d^2\alpha\end{aligned} \tag{2.15}$$

(Louisell, 1964, p. 246). The mean number of signal photons is now $N_s = |\mu|^2$.

This density operator ρ_1 does not commute with the density operator ρ_0 of Eq. (2.9), and there exists no set of simultaneous eigenstates. To describe and evaluate the optimum detector, it is necessary to diagonalize

[1] The signal is superposed on the noise in the manner outlined by Glauber (1963, p. 2778). The noise can be thought of as having been turned on first, bringing the system to a mixture of states described by ρ_0 of Eq. (2.9), after which the signal is turned on and the mixture described by ρ_1 results.

the operator $\rho_1 - \lambda\rho_0$ for arbitrary values of λ between 0 and ∞, and this appears to be a formidable problem.

The matrix elements $\rho_{1nm} = \langle n|\,\rho_1\,|m\rangle$ of the density operator ρ_1 in the number representation can be obtained from the generating function

$$\begin{aligned} R_1(\alpha^*, \beta) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\rho_{1nm}(\alpha^*)^n\beta^m}{\sqrt{n!\,m!}} \\ &= (1 - v_0)\exp\,[v_0\,\alpha^*\beta + (1 - v_0)(\alpha^*\mu + \beta\mu^* - |\,\mu\,|^2)]. \end{aligned} \tag{2.16}$$

In Glauber's representation the eigenstates $|\eta_k\rangle$ of the operator $\rho_1 - \lambda\rho_0$ might be determined by solving the integral equation

$$\int [R_1(\alpha^*, \beta) - \lambda R_0(\alpha^*, \beta)]e^{-|\beta|^2}F_k(\beta^*)\,d^2\beta/\pi = \eta_k\,F_k(\alpha^*), \tag{2.17}$$

where $R_0(\alpha^*, \beta)$ is obtained by setting $\mu = 0$ in Eq. (2.16), and

$$\begin{aligned} F_k(\alpha^*) &= \langle\alpha\,|\,\eta_k\rangle\,e^{|\alpha|^2/2}, \\ |\,\eta_k\rangle &= \int |\,\alpha\rangle\,F_k(\alpha^*)e^{-|\alpha|^2/2}\,d^2\alpha/\pi. \end{aligned} \tag{2.18}$$

Solving the integral equation (2.17) is equivalent to diagonalizing the infinite matrix $\rho_{1nm} - \lambda\rho_{0nm}$, $\rho_{0nm} = (1 - v_0)v_0^m\delta_{nm}$.

(iv) *A Coherent Signal of Unknown Phase.* It is unlikely that the receiver will know in advance the phase of a coherent optical signal, for a shift in the relative positions of transmitter and receiver through a distance of the order of a wavelength will alter the phase by a considerable fraction of 2π. When the phase of the complex parameter μ in Eq. (2.15) is completely unknown, it should be assigned the least favorable prior distribution, which is uniform over the interval $(0, 2\pi)$. The density operator under hypothesis H_1 is then obtained by averaging Eq. (2.15) with respect to that distribution of $\arg\mu = \psi$:

$$\begin{aligned} \rho_1 &= (\pi N)^{-1}\int_0^{2\pi}\frac{d\psi}{2\pi}\int \exp\,[-(|\,\alpha\,|^2 - 2\,|\,\alpha\,|\,|\,\mu\,|\cos(\psi - \theta) \\ &\qquad\qquad + |\,\mu\,|^2)/N]\cdot|\,\alpha\rangle\langle\alpha\,|\,d^2\alpha \\ &= (\pi N)^{-1}\int \exp\,[-(|\,\alpha\,|^2 + |\,\mu\,|^2)/N]I_0(2\,|\,\alpha\,|\,|\,\mu\,|/N)|\alpha\rangle\langle\alpha|\,d^2\alpha, \end{aligned} \tag{2.19}$$

where $\theta = \arg\alpha$ and $I_0(x)$ is the modified Bessel function. The density operator ρ_0 for the field in the absence of a signal is still given by Eq. (2.9).

Both density operators ρ_0 and ρ_1 are diagonal in the number representation, for the integrands in Eqs. (2.9), (2.19) depend on α only through its modulus $|\alpha|$. The observer can therefore base his decision about the presence or absence of a signal on a measurement of the number n of photons in the mode. Under hypothesis H_0 this number is distributed according to the geometrical distribution in Eq. (2.11). When a signal is present, the "Laguerre distribution" applies (Lachs, 1965),

$$\begin{aligned} P_{1k} &= (N+1)^{-1} \exp\left[-N_s/(N+1)\right] v_0^{\,k} L_k[-N_s/N(N+1)] \\ v_0 &= N/(N+1), \end{aligned} \tag{2.20}$$

where $L_k(x)$ is the kth Laguerre polynomial and $N_s = |\mu|^2$ is the average number of photons in the component of the field due to the signal.

The observer chooses hypothesis H_1 whenever the likelihood ratio $\exp[-N_s/(N+1)]L_k[-N_s/N(N+1)]$ exceeds a decision level λ. If n_0' is the smallest integer k for which this is possible, the false-alarm probability is $Q_0 = v_0^{n_0'}$. The detection probability

$$Q_d = \sum_{k=n_0'}^{\infty} P_{1k}$$

cannot apparently be expressed in any simple form.

The moments of the distribution in Eq. (2.20) can be obtained from the moment-generating function

$$\begin{aligned} \sum_{k=0}^{\infty} P_{1k} u^k &= (1 - v_0)(1 - v_0 u)^{-1} \\ &\quad \cdot \exp\left[-N_s(1 - v_0)(1 - u)(1 - v_0 u)^{-1}\right]. \end{aligned} \tag{2.21}$$

In particular the mean and the variance of n are

$$\mathbf{E}(n \mid H_1) = N + N_s, \qquad \mathrm{Var}_1\, n = N(N+1) + (2N+1)N_s. \tag{2.22}$$

When the expected number of photons in the mode is large, the distribution P_{1k} can be written in the approximate form (Erdélyi *et al.*, 1953, Vol. 1, p. 280, Eq. [6.13 (15)])

$$P_{1k} \cong N^{-1} \exp\left[-(k + N_s)/N\right] I_0(2\sqrt{N_s k}/N), \tag{2.23}$$

which is the noncentral Rayleigh or Rice distribution that appears in the theory of detecting a signal of unknown phase in the presence of

Gaussian random noise (Helstrom, 1960, Chapt. V). The power signal-to-noise ratio is then

$$d^2 \cong 2N_s/N, \tag{2.24}$$

which in the limit $\hbar\omega \ll K\mathfrak{I}$ becomes, by Eq. (2.12), $d^2 \doteq 2E/K\mathfrak{I}$, where $E = N_s\hbar\omega$ is the energy of the signal component of the field. When Eq. (2.23) applies, the probability of detection can be expressed approximately in terms of the Q-function,[2]

$$\begin{aligned} Q_d &\cong Q(\sqrt{2N_s/N}, \sqrt{2n_0/N}), \\ Q(\alpha, \beta) &= \int_\beta^\infty x \exp[-(x^2 + \alpha^2)/2] I_0(\alpha x)\, dx. \end{aligned} \tag{2.25}$$

At high frequencies, $\omega \gg K\mathfrak{I}/\hbar$, $v_0 = e^{-w} \ll 1$, and the Laguerre distribution becomes approximately the Poisson distribution for the number of signal photons in the mode,

$$P_{1k} = N_s^{\,k} \exp(-N_s)/k!, \tag{2.26}$$

as can be shown by keeping only the highest power in the Laguerre polynomial. If v_0 is less than the pre-assigned value of the false-alarm probability, $n_0' = 1$, and the receiver declares a signal present whenever it counts any photons at all in the mode. The probability of detection is less than 1 only because of the stochastic nature of the signal itself.

3. THE THRESHOLD RECEIVER

(i) *The "Classical" Threshold Receiver.* The likelihood-ratio receiver prescribed by signal-detection theory is sometimes difficult to implement and often suffers the disadvantage of depending on the amplitude of the signal, which may be unknown in advance. The designer may then turn to the "threshold receiver," which is a limiting form of the likelihood-ratio receiver obtained by letting the strength of the signal go to zero. If we denote by θ a parameter measuring the signal strength—it is usually taken as the square of the amplitude—, the threshold receiver is an embodiment of the statistic

$$g = \lim_{\theta \to \infty} \frac{\partial}{\partial \theta} \ln [p_1(\mathbf{x}; \theta)/p_0(\mathbf{x})], \tag{3.1}$$

where $\mathbf{x}$ denotes the set of measurements of the input to the receiver,

[2] As shown in Appendix B, the Q-function also provides a generating function for the cumulative Laguerre distribution.

$p_1(\mathbf{x}; \theta)$ is their joint probability density function (p.d.f.) when a signal of strength θ is present, and $p_0(\mathbf{x}) = p_1(\mathbf{x}; 0)$ is their joint p.d.f. under hypothesis H_0.

In quantum detection we have the additional problem of diagonalizing the operator $\rho_1 - \lambda\rho_0$ when the density operators ρ_0 and ρ_1 do not commute, and we should try to find out whether a threshold approximation can be used to determine a good detection statistic when the quantum-mechanical problem cannot be solved or when its solution is too complicated to implement.

Middleton (1966) has defined the best threshold receiver as the one that minimizes both the average cost $\bar{C}(\theta)$ and its first derivative $d\bar{C}/d\theta$ in the limit $\theta \to 0$. This average cost $\bar{C}(\theta)$ is given as in Eq. (1.6), in which the probability Q_d of detection and hence also the average cost are now functions of the signal strength θ. He has shown that when the p.d.f.'s $p_1(\mathbf{x}; \theta)$, $p_0(\mathbf{x})$ are continuous in $\mathbf{x}$ and θ, the derivative of the logarithm of the likelihood ratio, as in Eq. (3.1), provides the best threshold receiver. This approach is closed to us, however, for in quantum reception we must allow for the possibility that the quantities to be measured take on only discrete values. The average cost $\bar{C}(\theta)$ may then be a discontinuous function of the signal strength θ. The examples in parts (ii) and (iv) of Section 2 are cases in point.

The threshold receiver is most nearly optimum when the signals to be detected really are very weak. In order to attain a satisfactory combination of false-alarm and detection probabilities, however, it is then necessary for the signal to be repeated a number of times to give the receiver many independent opportunities to observe it. This might be done by exposing a set of empty cavities, one after another, to the external field during intervals when the signals, if sent, are expected to arrive. If the noise and the signal have the same distributions from one exposure to the next, a threshold detection operator is measured for each cavity, and the results for all the cavities are summed. If the sum exceeds a certain decision level, the observer decides that a signal is present.

When a receiver measures a large number M of identically distributed statistics $g_1, g_2, \cdots, g_M$ and bases its decision on their sum

$$G = g_1 + g_2 + \cdots + g_M, \tag{3.2}$$

the false-alarm and detection probabilities can be approximately determined by appealing to the central-limit theorem, which states that the larger the number M, the more nearly Gaussian are the distributions of the sum G under the two hypotheses. Those probabilities are then given by

$$\begin{aligned} Q_0 &\cong \operatorname{erfc} x, \qquad Q_d \cong 1 - \operatorname{erfc}(D\sqrt{M} - x), \\ D^2 &= [\mathbf{E}(g \mid H_1) - \mathbf{E}(g \mid H_0)]^2/\operatorname{Var}_0 g, \end{aligned} \tag{3.3}$$

where

$$\operatorname{erfc} x = (2\pi)^{-1/2} \int_x^\infty e^{-t^2/2}\, dt \tag{3.4}$$

is the error-function integral and $\operatorname{Var}_0 g$ is the variance of g under hypothesis H_0. We call the quantity D^2 the "equivalent signal-to-noise ratio" for a receiver that sums the statistic g, which may or may not be related to the likelihood ratio as in Eq. (3.1).

The performance of two such receivers summing statistics g' and g'' can conveniently be compared by fixing the false-alarm and detection probabilities and asking for the ratio M''/M' of the number of independent observations needed to attain them in the limit when both M' and M'' are large. This ratio is called the "asymptotic relative efficiency" (a.r.e.) of the receivers, and eq. (3.3) shows that it is given by

$$\text{a.r.e.} = \lim_{\theta \to \infty} D'^2/D''^2,$$

where D'^2 and D''^2 are the equivalent signal-to-noise ratios of the two receivers (Capon, 1961).

The receiver that shows up best in such a comparison is the one that has the greatest equivalent signal-to-noise ratio in the limit $\theta \to 0$, and in the domain of ordinary detection theory this receiver turns out to be the threshold receiver based on the statistic in Eq. (3.1). To show this we use the following argument. We suppose that for each exposure the receiver obtains a set $\mathbf{x}$ of measurements of its input and forms the statistic $f(\mathbf{x})$. With no loss of generality we may fix the mean value of this statistic under hypothesis H_0 at 0:

$$\mathbf{E}[f(\mathbf{x}) \mid H_0] = \int f(\mathbf{x}) p_0(\mathbf{x})\, d^n\mathbf{x} = 0, \tag{3.5}$$

as can be seen from eq. (3.3), for adding a constant to the statistic does not change the equivalent signal-to-noise ratio. Schwarz's inequality then shows that the equivalent signal-to-noise ratio is bounded by

$$D^2 = \left\{\int f(\mathbf{x})[p_1(\mathbf{x};\theta) - p_0(\mathbf{x})]\, d^n\mathbf{x}\right\}^2 \Big/ \int [f(\mathbf{x})]^2 p_0(\mathbf{x})\, d^n\mathbf{x}$$

$$= \frac{\left\{\int f(\mathbf{x})[p_0(\mathbf{x})]^{1/2}[p_1(\mathbf{x};\theta) - p_0(\mathbf{x})][p_0(\mathbf{x})]^{-1/2}\, d^n\mathbf{x}\right\}^2}{\int \{f(x)[p_0(x)]^{1/2}\}^2\, d^n\mathbf{x}} \tag{3.6}$$

$$\leqq \int [p_1(\mathbf{x};\theta) - p_0(\mathbf{x})]^2 [p_0(\mathbf{x})]^{-1}\, d^n\mathbf{x} = D^2_{\max}.$$

This maximum equivalent signal-to-noise ratio

$$D^2_{\max} = \int p_0(\mathbf{x}) \left[\frac{p_1(\mathbf{x};\theta)}{p_0(\mathbf{x})} - 1\right]^2 d^n\mathbf{x} \tag{3.7}$$

is attained by the statistic

$$f(\mathbf{x}) = \frac{p_1(\mathbf{x};\theta)}{p_0(\mathbf{x})} - 1, \tag{3.8}$$

which satisfies Eq. (3.5) (Rudnick, 1962).

At low signal strengths the statistic in Eq. (3.8) becomes

$$f(\mathbf{x}) \doteq \theta \left[\frac{\partial}{\partial\theta} \ln \frac{p_1(\mathbf{x};\theta)}{p_0(\mathbf{x})}\Big|_{\theta=0}\right] \tag{3.9}$$

and is proportional to the threshold statistic g of Eq. (3.1). The "threshold signal-to-noise ratio" D_θ^2 is then given by the limiting form of Eq. (3.7),

$$D_\theta^2 \doteq \theta^2 \int p_0(\mathbf{x}) \left[\frac{\partial}{\partial\theta} \ln \frac{p_1(\mathbf{x};\theta)}{p_0(\mathbf{x})}\Big|_{\theta=0}\right]^2 d^n\mathbf{x}, \tag{3.10}$$

as shown by Capon (1961), who termed the ratio D_θ^2/θ^2 the "efficacy" of the threshold receiver.

(ii) *The Quantum-Mechanical Counterpart.* By analogy we define the quantum-mechanical threshold receiver as the limiting form, as θ goes to 0, of one embodying a detection operator Π_θ for which the equivalent signal-to-noise ratio D^2 given by

$$D^2 = \frac{[\mathrm{Tr}\,(\rho_1 \Pi_\theta) - \mathrm{Tr}\,(\rho_0 \Pi_\theta)]^2}{\mathrm{Tr}\,(\rho_0 \Pi_\theta^2) - [\mathrm{Tr}\,(\rho_0 \Pi_\theta)]^2} \tag{3.11}$$

is largest. Such a receiver will be best on the basis of asymptotic relative efficiency. We shall now derive the form of the statistic Π_θ and pass to the limit $\theta \to 0$.

Again we lose no generality by fixing the expected value of the statistic under hypothesis H_0 at zero,

$$\mathrm{Tr}\,(\rho_0 \Pi_\theta) = 0. \tag{3.12}$$

We can then maximize D^2 by maximizing $[\mathrm{Tr}\,(\rho_1 \Pi_\theta)]^2$ for a fixed value of $\mathrm{Tr}\,(\rho_0 \Pi_\theta^2)$, for if Π_θ maximizes D^2, so does $C\Pi_\theta$ for any constant C.

In a matrix representation we introduce Lagrange multipliers u and v and maximize

$$\begin{aligned}[]&[\mathrm{Tr}\,(\rho_1 \Pi_\theta)]^2 - u\,\mathrm{Tr}\,(\rho_0 \Pi_\theta^2) - v\,\mathrm{Tr}\,(\rho_0 \Pi_\theta) \\ &\quad = \Big(\sum_{m,n} \rho_{1mn} \Pi_{\theta nm}\Big)^2 - u \sum_{m,n,r} \rho_{0mn} \Pi_{\theta nr} \Pi_{\theta rm} - v \sum_{m,n} \rho_{0mn} \Pi_{\theta nm}\,,\end{aligned} \tag{3.13}$$

which leads by differentiation to the equations

$$2\rho_{1mn}\,\mathrm{Tr}\,(\rho_1 \Pi_\theta) - u \sum_s \rho_{0sn} \Pi_{\theta ms} - u \sum_s \rho_{0ms} \Pi_{\theta sn} - v\rho_{0mn} = 0 \tag{3.14}$$

or in operator notation

$$2\rho_1\,\mathrm{Tr}\,(\rho_1 \Pi_\theta) = u(\Pi_\theta \rho_0 + \rho_0 \Pi_\theta) + v\rho_0\,. \tag{3.15}$$

Within an unimportant constant of proportionality the operator Π_θ is thus the solution of the equation

$$2(\rho_1 - \rho_0) = \Pi_\theta \rho_0 + \rho_0 \Pi_\theta\,. \tag{3.16}$$

Since $\mathrm{Tr}\,\rho_0 = \mathrm{Tr}\,\rho_1 = 1$, Eq. (3.12) is satisfied. The maximum equivalent signal-to-noise ratio $D^2_{\max}$ is

$$D^2_{\max} = \mathrm{Tr}\,(\rho_0 \Pi_\theta^2) = \mathrm{Tr}\,[(\rho_1 - \rho_0)\Pi_\theta] = \mathrm{Tr}\,(\rho_1 \Pi_\theta). \tag{3.17}$$

If the density operator $\rho_1(\theta)$ is differentiable with respect to θ, the threshold detection operator Π_0 can be obtained from Eq. (3.16) as the solution of the equation

$$2\left.\frac{\partial \rho_1}{\partial \theta}\right|_{\theta=0} = \Pi_0\,\rho_0 + \rho_0\,\Pi_0\,, \tag{3.18}$$

and the threshold signal-to-noise ratio D_θ^2 is

$$D_\theta^2 \doteq \theta^2\,\mathrm{Tr}\,(\rho_0\,\Pi_0^2) = \theta^2\,\mathrm{Tr}\left[\left.\frac{\partial \rho_1}{\partial \theta}\right|_{\theta=0} \Pi_0\right]. \tag{3.19}$$

If we have M chances to observe the signals by exposing M independent cavities to the external field, the dynamical variable associated with the operator Π_0 is measured for the field in each cavity, and the results are added. If the sum exceeds a certain decision level, it is decided that a signal or train of signals was present in the external field. If the number M of exposures is large, the false-alarm and detection probabilities will again be given by Eq. (3.3), with the signal-to-noise ratio D^2 taken from Eq. (3.19).

(iii) *Threshold Detection of a Coherent Signal.* As an example we return to the detection of a coherent signal of known phase by measurement of a single mode of a cavity, discussed in part (iii) of Section 2. When the amplitude parameter $|\mu|$ is small, the density operator ρ_1 in Eq. (2.15) is approximately

$$\begin{aligned} \rho_1 &\doteq [1 + (e^w - 1)\mu^* a + (1 - e^{-w})\mu a^+]\rho_0 \\ &= \rho_0[1 + (1 - e^{-w})\mu^* a + (e^w - 1)\mu a^+], \end{aligned} \tag{3.20}$$

where ρ_0 is again given by Eq. (2.9). These forms can be shown to be equivalent by using the operational rules (Louisell, 1964, p. 111)

$$e^{wn} a e^{-wn} = a e^{-w}, \; e^{wn} a^+ e^{-wn} = e^w a^+, \tag{3.21}$$

where again $n = a^+ a$ is the number operator. They can be obtained by expanding the exponential in the integral form of ρ_1 in Eq. (2.15) and using Eq. (2.6) and its adjoint. The same rules can be used to show that the Hermitian operator

$$\Pi_0 = (\mu a^+ + \mu^* a)/|\mu| \tag{3.22}$$

satisfies Eq. (3.18) when θ is taken as $|\mu|$ and the derivative is obtained from Eq. (3.20). It is independent of the amplitude $|\mu|$ of the signal. Since Π_0 commutes with neither ρ_0 nor ρ_1, and hence not with $\rho_1 - \lambda\rho_0$ for any λ, it is not an optimum detection operator in the sense of Section 2.

A measurement of the statistic Π_0 yields a result that is Gaussian distributed with expected values given by

$$\mathbf{E}(\Pi_0 \mid H_0) = 0, \qquad \mathbf{E}(\Pi_0 \mid H_1) = 2|\mu| \tag{3.23}$$

and with variance equal to $(2N + 1)$ under both hypotheses, where as before N is the average number of photons in the mode due to the background radiation (Louisell, 1964, p. 247). The probability of detection and the false-alarm probability in M independent trials are given exactly by Eq. (3.3), in which the signal-to-noise ratio D^2 is now

$$D^2 = 4N_s/(2N + 1), \tag{3.24}$$

where $N_s = |\mu|^2$ is the mean number of signal photons. Here it is unnecessary to call on the central-limit theorem.

If for N we substitute the Planck formula, Eq. (2.12), and if we put $N_s = E/\hbar\omega$, where E is the energy of the signal component of the field, the equivalent signal-to-noise ratio D^2 is

$$D^2 = \frac{4E}{\hbar\omega} \tanh (\hbar\omega/2K\mathfrak{T}).$$

For $\hbar\omega \ll K\mathfrak{T}$ this reduces to $D^2 \doteq 2E/K\mathfrak{T}$, the usual signal-to-noise ratio for the detection of a coherent signal in Gaussian thermal noise of equivalent absolute temperature $\mathfrak{T}$. At high frequencies, $D^2 \doteq 4N_s = 4E/\hbar\omega$. The detectability of the signal now depends only on its own statistical properties.

4. MULTIMODE DETECTION OF A COHERENT SIGNAL

(i) *The Density Operators.* When the aperture of the ideal receiver is opened, it must be expected that the interaction with the external field will excite many modes of the cavity. Measurements of all these modes must then be made in order to exhaust the information available for deciding about the presence or absence of a signal in the external field. We shall now see what measurements the theory prescribes for detecting a coherent signal of random phase in the presence of thermal background radiation. By necessity our treatment is limited to the threshold receiver.

Quantum mechanics represents the electric field at point $\mathbf{r}$ in the cavity and at time t by an operator $\boldsymbol{\varepsilon}(\mathbf{r}, t)$ that can be decomposed into its positive- and negative-frequency parts,

$$\boldsymbol{\varepsilon}(\mathbf{r}, t) = \boldsymbol{\varepsilon}^{(+)}(\mathbf{r}, t) + \boldsymbol{\varepsilon}^{(-)}(\mathbf{r}, t), \qquad \boldsymbol{\varepsilon}^{(-)}(\mathbf{r}, t) = [\boldsymbol{\varepsilon}^{(+)}(\mathbf{r}, t)]^+, \tag{4.1}$$

the one part being the adjoint of the other. In terms of the mode eigenfunctions $\mathbf{u}_{\mathrm{m}}(\mathbf{r})$, which are solutions of the Helmholtz equation with proper boundary conditions at the walls of the cavity, the positive-frequency part of the field operator is written as

$$\boldsymbol{\varepsilon}^{(+)}(\mathbf{r}, t) = i \sum_{\mathrm{m}} (\hbar\omega_{\mathrm{m}}/2)^{1/2} a_{\mathrm{m}} \mathbf{u}_{\mathrm{m}}(\mathbf{r}) \exp(-i\omega_{\mathrm{m}} t), \tag{4.2}$$

where a_{m} is the annihilation operator and ω_{m} the eigenfrequency of mode

$\mathbf{m}$. The operator $a_{\mathbf{m}}$ and its adjoint $a_{\mathbf{m}}^+$ obey the commutation rules given for annihilation and creation operators in Eq. (2.2), and they possess the same array of eigenstates as described there. In particular the operator $n_{\mathbf{m}} = a_{\mathbf{m}}^+ a_{\mathbf{m}}$ corresponds to the number of photons in mode $\mathbf{m}$. The operators for different modes commute. The mode functions $\mathbf{u}_{\mathbf{m}}(\mathbf{r})$ are orthonormal over the volume of the cavity.

The index $\mathbf{m}$ is in general a set of four numbers. For a rectangular cavity, for instance, with periodic boundary conditions at the walls, the mode functions are of the form (Louisell, 1964, p. 153)

$$\begin{aligned} \mathbf{u}_{\mathbf{m}}(\mathbf{r}) &= V^{-1/2}\mathbf{e}_\delta \exp\left[-i(m_1 x + m_2 y + m_3 z)\right] \\ m_i &= 2\pi n_i/L_i, \qquad i = 1, 2, 3; \qquad \delta = 1, 2, \end{aligned} \tag{4.3}$$

where L_1, L_2, L_3 are the lengths of the sides of the cavity, $V = L_1 L_2 L_3$ is its volume, n_1, n_2, n_3 are integers, and $\mathbf{e}_1$, $\mathbf{e}_2$ are unit vectors perpendicular to each other and to the propagation vector (m_1, m_2, m_3). Here $\mathbf{m}$ can be taken as the quadruple (m_1, m_2, m_3, δ), with δ specifying the polarization of the mode.

The electric field outside the cavity can be represented in the same way if, as is usually done for mathematical convenience, the external region is imagined to be enclosed in a huge box. The quantum-mechanical counterpart of a coherent signal outside the cavity is a field in a so-called "coherent state," which is simultaneously a right-eigenstate of the annihilation operators for the modes of that field. Such a coherent state will be created by a classical current distribution, one that suffers no unpredictable reaction from the electromagnetic field. A mixture of coherent states whose sets of complex mode amplitudes differ only in over-all phase, to which a uniform distribution over $(0, 2\pi)$ is assigned, may be a good model of the radiation from a carefully controlled laser oscillator (Glauber, 1965). The task of the receiver is to decide whether such a coherent field of random phase is present or not during a certain interval.

The background radiation outside the cavity will be assumed to be of the Gaussian thermal kind that has been so extensively treated (Glauber, 1963; Keller, 1965). In the absence of a signal the modes of the external field are described by a density operator that is a product of density operators of the type shown in Eq. (2.9), one for each mode of the external field. The mean number of photons in each mode is related in thermal equilibrium to the frequency of the mode by the Planck law, Eq. (2.12). When a signal is present, the density operator of the external field is a product of operators of the form given in Eq. (2.15).

We suppose that the cavity of the ideal receiver is initially empty. When the aperture is open during the interval $(0, T)$, the field inside the cavity is coupled to the external field, and its modes make transitions into various excited states. If the aperture remains open for a time T much greater than the periods of any of the oscillations contained in the signal, and if the opening and closing of the aperture are controlled so that no transitions are induced in the electromagnetic field, the density operators ρ_0 and ρ_1 of the field within the cavity after the aperture is closed will to a good approximation be of the same Gaussian type as those initially describing the external field (Helstrom, 1966). That is, if there was no signal present externally (hypothesis H_0), the density operator for the field in the cavity is of the form

$$\rho_0 = \pi^{-\nu} \,|\det \varphi\,|^{-1} \int \cdots \int \exp\Big[-\sum_{\mathrm{m}} \sum_{\mathrm{n}} \alpha_{\mathrm{m}}^*(\varphi^{-1})_{\mathrm{mn}}\, \alpha_{\mathrm{n}}\Big] \cdot \prod_{\mathrm{m}} |\,\alpha_{\mathrm{m}}\rangle\langle\alpha_{\mathrm{m}}\,|\, d^2\alpha_{\mathrm{m}}\,, \tag{4.4}$$

where ν is the number of modes in the cavity, assumed for the time being to be finite, and φ is a $\nu \times \nu$ mode correlation matrix:

$$\varphi_{km} = \mathrm{Tr}\,(\rho_0 a_m^{+} a_k). \tag{4.5}$$

If φ is diagonal, its diagonal elements are the average numbers of photons in each mode due to the thermal radiation; and if the aperture is open long enough for thermal equilibrium to be reached, these diagonal elements are given by the Planck law, Eq. (2.12), in terms of the frequencies of the modes of the cavity. The coherent state $\prod_{\mathrm{m}} |\alpha_{\mathrm{m}}\rangle$ is a simultaneous right-eigenstate of the annihilation operators a_{m} for the modes of the cavity, and Eq. (4.4) is simply a generalization of the density operator given in Eq. (2.9).

If there was a coherent signal present in the external field (hypothesis H_1), the density operator for the cavity field after the aperture was closed has the form

$$\rho_1 = \pi^{-\nu} \,|\det \varphi\,|^{-1} \int \cdots \int \cdot \exp\Big[-\sum_{\mathrm{m}} \sum_{\mathrm{n}} (\alpha_{\mathrm{m}}^* - \mu_{\mathrm{m}}^*)(\varphi^{-1})_{\mathrm{mn}}(\alpha_{\mathrm{n}} - \mu_{\mathrm{n}})\Big] \prod_{\mathrm{m}} |\,\alpha_{\mathrm{m}}\rangle\langle\alpha_{\mathrm{m}}\,|\, d^2\alpha_{\mathrm{m}}\,, \tag{4.6}$$

where μ_{m} is the amplitude of the coherent part of the field in mode **m**. To these amplitudes μ_{m} we shall later attach a common phase factor $e^{i\psi}$;

the resulting density operator will be denoted by $\rho_1(\psi)$. The observer must choose between the density operators ρ_0 and $\overline{\rho_1(\psi)}$, the bar indicating an average over the phase ψ, taken as unknown and uniformly distributed over the interval $(0, 2\pi)$. For the present, however, we can disregard this matter of phase.

Both the set $\{\mu_m\}$ of signal amplitudes and the mode correlation matrix φ can be determined from the signal field outside the cavity and the spectral distribution of the external background radiation. Under the assumptions mentioned earlier that the interval $(0, T)$ is much longer than the periods of any signal frequencies and that the opening and closing of the aperture do not affect the fields, the density operator of the internal field can be expressed in terms of the interaction between the cavity and the external field as described by classical electromagnetic theory. It is only necessary to know the coupling coefficients between the internal and the external modes, which specify a linear transformation from the external to the internal signal field and a bilinear transformation of the mode correlation matrices:

$$\begin{aligned} \mu_m &= \sum_n Z_{mn}\mu_n^{\;0} \\ \varphi_{mn} &= \sum_k \sum_j Z_{mk}\varphi^0_{kj}Z^*_{nj}\,, \end{aligned} \tag{4.7}$$

where the superscripts 0 refer to the external field before the aperture was opened. The coefficients Z_{mn} will be functions of the duration T of the interval in which the aperture is open. If the cavity is not initially empty, but is filled with thermal radiation, the expression for φ_{mn} will contain additional terms describing the effect on that internal radiation of the interaction with the external field.

(ii) *The Threshold Receiver*. When more than one mode of the field contains a component due to the signal, the density operators ρ_0 and $\overline{\rho_1(\psi)}$ do not commute, and no way of diagonalizing the operator $\overline{\rho_1(\psi)} - \lambda\rho_0$ has been found. We are therefore reduced to employing a threshold detection operator Π_θ of the kind defined in Section 3. For this we solve Eq. (3.16) under the approximation that the signal is weak.

The calculation is simplest in Glauber's representation, and we therefore write the equation as

$$\begin{aligned} 2\langle\alpha\,|(\rho_1 - \rho_0)|\,\gamma\rangle = \int P_0(\beta)[\langle\alpha\mid\beta\rangle\langle\beta\mid\Pi_\theta\mid\gamma\rangle \\ + \langle\alpha\mid\Pi_\theta\mid\beta\rangle\langle\beta\mid\gamma\rangle]\prod_m d^2\beta_m\,, \end{aligned} \tag{4.8}$$

where we have let α stand for the whole set of complex amplitudes α_{m}, β for the set of β_{m}'s, and so on, to simplify the appearance of the equations. Here

$$P_0(\beta) = \pi^{-\nu} \mid \det \varphi \mid^{-1} \exp \left[-\sum_{\mathrm{m}} \sum_{\mathrm{n}} \beta_{\mathrm{m}}^*(\varphi^{-1})_{\mathrm{mn}}\beta_{\mathrm{n}}\right].$$

Following Glauber (1963) we define the functionals

$$\begin{aligned} R_i(\alpha^*, \beta) &= \langle \alpha \mid \rho_i \mid \beta \rangle \exp \tfrac{1}{2} \sum_{\mathrm{m}} (\mid \alpha_{\mathrm{m}} \mid^2 + \mid \beta_{\mathrm{m}} \mid^2) \\ Q(\alpha^*, \beta) &= \langle \alpha \mid \Pi_\theta \mid \beta \rangle \exp \tfrac{1}{2} \sum_{\mathrm{m}} (\mid \alpha_{\mathrm{m}} \mid^2 + \mid \beta_{\mathrm{m}} \mid^2) \end{aligned} \tag{4.9}$$

and use (Glauber, 1963, Eq. (3.32), p. 2771)

$$\langle \alpha \mid \beta \rangle = \exp \left[\sum_{\mathrm{m}} (\alpha_{\mathrm{m}}^*\beta_{\mathrm{m}} - \tfrac{1}{2} \mid \alpha_{\mathrm{m}} \mid^2 - \tfrac{1}{2} \mid \beta_{\mathrm{m}} \mid^2)\right] \tag{4.10}$$

to write Eq. (4.8) in the form

$$\begin{aligned} 2[R_1(\alpha^*, \gamma) - R_0(\alpha^*, \gamma)] &= \int \cdots \int P_0(\beta) \exp \left(-\sum_{\mathrm{m}} \mid \beta_m \mid^2\right) \\ \cdot \left[\exp \left(\sum_{\mathrm{m}} \alpha_{\mathrm{m}}^*\beta_{\mathrm{m}}\right) Q(\beta^*, \gamma) \right. &+ \left. \exp \left(\sum_{\mathrm{m}} \beta_{\mathrm{m}}^*\gamma_{\mathrm{m}}\right) Q(\alpha^*, \beta)\right] \prod_{\mathrm{m}} d^2\beta_{\mathrm{m}}. \end{aligned} \tag{4.11}$$

This integral equation must be solved to determine the operator Π_θ.

From Eqs. (4.4), (4.6), (4.9) it can be shown by evaluating a multidimensional Gaussian integral that the functional corresponding to the operator $\rho_1(\psi)$ is

$$\begin{aligned} R_1(\alpha^*, \beta; \psi) = \mid \det (\mathbf{I} + \varphi)\mid^{-1} \exp [&-\mathbf{M}^+(\mathbf{I} + \varphi)^{-1}\mathbf{M} \\ &+ \mathbf{A}^+(\mathbf{I} + \varphi)^{-1}\mathbf{M}e^{i\psi} + e^{-i\psi}\mathbf{M}^+(\mathbf{I} + \varphi)^{-1}\mathbf{B} \\ &+ \mathbf{A}^+(\mathbf{I} + \varphi^{-1})^{-1}\mathbf{B}], \end{aligned} \tag{4.12}$$

where $\mathbf{A}^+$ is the row vector of the α_{k}^*'s, $\mathbf{B}$ the column vector of the β_{k}'s, $\mathbf{M}$ the column vector of the μ_{k}'s, and $\mathbf{I}$ the $\nu \times \nu$ identity matrix. (Eq. (2.16) gives the one-dimensional form of this functional.) The functional $R_0(\alpha^*, \beta)$ is obtained by setting $\mathbf{M} = 0$ in Eq. (4.12).

The threshold operator is found by solving Eq. (3.16) under the assumption that the signal strength is small. We therefore expand Eq. (4.12) in powers of the signal strength and average the phase ψ over the interval $(0, 2\pi)$ to obtain the approximate functional corresponding to $\overline{\rho_1(\psi)}$:

$$\overline{R_1(\alpha^*, \beta; \psi)} \doteq R_0(\alpha^*, \beta)[1 - \mathbf{M}^+(\mathbf{I} + \varphi)^{-1}\mathbf{M} + \mathbf{A}^+\mathbf{K}\mathbf{B}], \tag{4.13}$$

where $\mathbf{K}$ is the matrix

$$\mathbf{K} = (\mathbf{I} + \varphi)^{-1}\mathbf{M}\mathbf{M}^{+}(\mathbf{I} + \varphi)^{-1}. \tag{4.14}$$

As a trial solution we take the detection operator as the Hermitian quadratic form

$$\Pi_\theta = b + \tfrac{1}{2}\sum_{k,m} q_{km}(a_k{}^{+}a_m + a_m a_k{}^{+}) = b' + \sum_{k,m} q_{km}a_k{}^{+}a_m, \tag{4.15}$$

where b and b' are constants related, because of Eq. (2.2), by

$$b' = b + \tfrac{1}{2}\operatorname{Tr}\mathbf{Q}, \qquad \mathbf{Q} = \| q_{km} \|. \tag{4.16}$$

For this operator, from Eq. (4.9),

$$Q(\alpha^*, \beta) = (b' + \mathbf{A}^{+}\mathbf{Q}\mathbf{B}) \exp(\mathbf{A}^{+}\mathbf{B}) \tag{4.17}$$

(Glauber, 1963, Eq. (5.11), p. 2774). Putting this into Eq. (4.11) and integrating we find, by Eq. (4.13),

$$\begin{aligned} &2R_0(\alpha^*, \beta)[\mathbf{A}^{+}\mathbf{K}\mathbf{B} - \mathbf{M}^{+}(\mathbf{I} + \varphi)^{-1}\mathbf{M}] \\ &\quad = R_0(\alpha^*, \beta)\{b' + \mathbf{A}^{+}[(\mathbf{I} + \varphi^{-1})^{-1}\mathbf{Q} + \mathbf{Q}(\mathbf{I} + \varphi^{-1})^{-1}]\mathbf{B}\} \end{aligned} \tag{4.18}$$

whence the constant b' is given by

$$b' = -\mathbf{M}^{+}(\mathbf{I} + \varphi)^{-1}\mathbf{M}, \tag{4.19}$$

and the matrix $\mathbf{Q}$ specifying the threshold detection operator is the solution of the equation

$$2\mathbf{K} = (\mathbf{I} + \varphi^{-1})^{-1}\mathbf{Q} + \mathbf{Q}(\mathbf{I} + \varphi^{-1})^{-1}$$

or

$$2\mathbf{M}\mathbf{M}^{+} = \varphi\mathbf{Q}(\mathbf{I} + \varphi) + (\mathbf{I} + \varphi)\mathbf{Q}\varphi. \tag{4.20}$$

Unitary transformations of the set of operators a_k correspond to setting up new mode structures that at a given point of time can serve to specify the field in the cavity as well as the expansion in Eq. (4.2) did. The annihilation operators for these modes obey the same commutation rules as the original set $\{a_k\}$. Gaussian density operators such as the one in Eq. (4.6) are transformed into new density operators of the same type, and the complex amplitudes α_k entering them transform in the same way as the operators a_k. It is thus possible to adopt a set of modes for which the mode correlation matrix takes on the diagonal form $\varphi = \| N_k{}'\delta_{km} \|$. In this representation, in which the signal amplitudes are $\mu_m{}'$, the elements of the matrix $\mathbf{Q}$ specifying the threshold detection operators are

$$Q_{\mathrm{km}} = [\tfrac{1}{2}(N_{\mathrm{k}}' + N_{\mathrm{m}}') + N_{\mathrm{k}}'N_{\mathrm{m}}']^{-1}\mu_{\mathrm{k}}'\mu_{\mathrm{m}}'^{*}. \tag{4.21}$$

Using this and Eq. (4.19) one can show that Eq. (3.12) is satisfied,

$$\mathrm{Tr}\,(\rho_0\Pi_\theta) = b' + \mathrm{Tr}\,(\mathbf{Q}\varphi) = b' + \sum_{\mathrm{k}} (N_k' + 1)^{-1}\,|\,\mu_{\mathrm{k}}'\,|^2 = 0. \tag{4.22}$$

The constant b' merely serves to make the expected outcome of the measurement of the detection operator Π_θ vanish when no signal is present.

In this representation the threshold detection operator is

$$\Pi_\theta = b' + \sum_{k,m} [\tfrac{1}{2}(N_{\mathrm{k}}' + N_{\mathrm{m}}') + N_{\mathrm{k}}'N_{\mathrm{m}}']^{-1}(\mu_{\mathrm{k}}'a_{\mathrm{k}}^{+})(\mu_{\mathrm{m}}'^{*}a_{\mathrm{m}}), \tag{4.23}$$

within an unimportant constant factor. At low frequencies the N_{k}''s are much greater than 1, and the operators a_{k} become classical mode amplitudes for the field in the cavity. One then effectively measures the quantity

$$|\sum_{\mathrm{m}} \mu_{\mathrm{m}}'^{*}a_{\mathrm{m}}/N_{\mathrm{m}}'\,|^2,$$

whose form corresponds to the squared output of a matched filter for detecting a signal in colored Gaussian noise (Helstrom, 1960, p. 147).

(iii) *The Performance of the Threshold Receiver.* To calculate the false-alarm and detection probabilities, the probability distributions of the outcome of a measurement of the operator in Eq. (4.23) are needed, and these are difficult to obtain. They can be worked out approximately from Edgeworth series, whose coefficients involve the cumulants of the distribution. The cumulants can be calculated by expanding in powers of u the logarithms of the moment-generating functions $\mathbf{E}[e^{uQ'}\mid H_i] = \mathrm{Tr}\,(\rho_i e^{uQ'})$, $i = 0, 1$, where Q' is the operator

$$Q' = \sum_{\mathrm{k,m}} q_{\mathrm{km}}a_{\mathrm{k}}^{+}a_{\mathrm{m}} = \Pi_\theta - b'.$$

Under hypothesis H_1 this moment-generating function is

$$\begin{aligned}\mathrm{Tr}\,(\rho_1 e^{uQ'}) &= |\det(\mathbf{I} - \varphi\mathbf{P})|^{-1}\exp \mathbf{M}^{+}\mathbf{P}(\mathbf{I} - \varphi\mathbf{P})^{-1}\mathbf{M} \\ &= \exp[\mathbf{M}^{+}\mathbf{P}(\mathbf{I} - \varphi\mathbf{P})^{-1}\mathbf{M} - \mathrm{Tr}\ln(\mathbf{I} - \mathbf{P}\varphi)], \end{aligned} \tag{4.24}$$

$$\mathbf{P} = \exp(u\mathbf{Q}) - \mathbf{I},$$

where the functions of matrices used here are defined in terms of their power-series expansions. The moment-generating function under hypothesis H_0 is obtained by setting the elements of $\mathbf{M}$ equal to 0.

If it is possible to expose a number M of receiver cavities to the external field independently, the false-alarm and detection probabilities can, when M is large, be determined approximately by Eq. (3.3). The equivalent signal-to-noise ratio for this threshold detector is, according to Eq. (3.17),

$$\begin{aligned} \eta^2 &= \mathrm{Tr}\,[(\rho_1 - \rho_0)\Pi_\theta) = \mathbf{M}^+\mathbf{Q}\mathbf{M} \\ &= \sum_{\mathbf{k},\mathbf{m}} [\tfrac{1}{2}(N_{\mathbf{k}}' + N_{\mathbf{m}}') + N_{\mathbf{k}}'N_{\mathbf{m}}']^{-1} \,|\, \mu_{\mathbf{k}} \,|^2 \,|\, \mu_{\mathbf{m}} \,|^2. \end{aligned} \tag{4.25}$$

If the aperture is kept open long enough for the modes to be nearly in thermal equilibrium, the $N_{\mathbf{k}}'$'s will be given approximately by the Planck law in terms of the frequencies of the modes. If the signal amplitudes $|\, \mu_{\mathbf{k}}' \,|$ are significant only over a range of frequencies $\omega_{\mathbf{k}}$ for which the $N_{\mathbf{k}}'$'s are nearly equal, the threshold signal-to-noise ratio is approximately equal to

$$D^2 = N_s^2/N(N+1), \tag{4.26}$$

where $N_s = \Sigma_{\mathbf{k}} \,|\, \mu_{\mathbf{k}} \,|^2$ is the total mean number of photons in the signal field of the cavity, and N is given by the Planck law, Eq. (2.12), for the signal carrier frequency ω. This is the same threshold signal-to-noise ratio as for the detector derived in Section 2, part (iv).

At low frequencies the signal-to-noise ratio reduces to $D^2 = (E/K\mathfrak{I})^2$, where E is the signal energy, K is Boltzmann's constant, and $\mathfrak{I}$ is the effective absolute temperature of the cavity. A threshold receiver for detecting classical signals of random phase in Gaussian thermal noise by summing the quadratically detected outputs of a matched filter is governed by the same threshold signal-to-noise ratio (Helstrom, 1960, p. 182). At high frequencies, $\omega \gg K\mathfrak{I}/\hbar$, on the other hand, if the only noise is thermal radiation, the mean number N of noise photons is much less than 1, and the effective signal-to-noise ratio is $D^2 \cong N_s^2/N$, which is characteristic of detectors in which Poisson-distributed numbers of photons are counted.

5. THRESHOLD RECEPTION OF AN INCOHERENT SIGNAL

Natural sources of light are made up of a very large number of independently radiating atoms. The light they emit can to a good approximation be described as an electromagnetic field in a mixture of states specified by a density operator of the Gaussian form in Eq. (4.4). That this is so follows from the same kind of argument that leads to a Gaussian distribution for the sum of a large number of independent random variables (Glauber, 1963, p. 2780). The mode correlation matrix appearing in the density operator is diagonal, and the diagonal element $\varphi_{\mathbf{kk}}$ associated with mode $\mathbf{k}$ of frequency $\omega_{\mathbf{k}}$ is given by

$$\varphi_{\mathbf{kk}} = \hbar^{-1} c^2 (\omega_{\mathbf{k}}/2\pi)^{-3} \Phi(\omega_{\mathbf{k}}\,;\, \boldsymbol{\theta}_{\mathbf{k}}), \tag{5.1}$$

where $\Phi(\omega;\, \boldsymbol{\theta})\, d\omega\, d\Omega$ is the radiant flux (erg/cm^2·sec) in a frequency range $(\omega,\, \omega + d\omega)$ having its direction in a solid angle $d\Omega$ about the unit vector $\boldsymbol{\theta}$. Here $\boldsymbol{\theta}_{\mathbf{k}}$ is the direction of propagation of mode $\mathbf{k}$, whose field is taken as a wave of the form given in Eq. (4.3), and c is the velocity of light. We call such a field of radiation an "incoherent signal" when it is generated by a source that is being turned on and off at regular intervals in accordance, let us say, with the 1's and 0's of a binary message.

When such an incoherent signal is superimposed on a field of thermal radiation, the combination is described by a density operator of the same Gaussian type. With the signal and the background independent, their mode correlation matrices, or equivalently their spectral densities, are simply additive (see footnote 1). To detect the incoherent signal, the aperture of our ideal receiver will be oriented toward its source and opened for an interval $(0,\, T)$ during which the signal, if transmitted, isexpected to arrive.

As before, the cavity is initially empty. After the aperture is closed its field will be in a mixture of states described by one of two density operators, ρ_0 and ρ_1, depending on whether the signal is absent or present. Under the assumption that the aperture is open for an interval much longer than the periods of oscillation of the field of the signal, the density operators will be of the same Gaussian form as in the external field,

$$\rho_i = \pi^{-\nu} \,|\det \varphi_i\,|^{-1} \int \cdots \int \exp\,[-\sum_{\mathbf{m}} \sum_{\mathbf{n}} \alpha_{\mathbf{m}}^{*} (\varphi_i^{-1})_{\mathbf{mn}}\, \alpha_{\mathbf{n}}]\, |\, \alpha_{\mathbf{m}}\rangle\langle \alpha_{\mathbf{m}} \prod_{\mathbf{m}} d^2\alpha_{\mathbf{m}}$$

$$\varphi_1 = \varphi_0 + \varphi_s\,, \qquad i = 0,\, 1, \tag{5.2}$$

where φ_0 and φ_s are the mode correlation matrices of the cavity fields resulting from the thermal background radiation and the signal, respectively. These can be obtained from the spectral distributions of the background and the signal outside the cavity by equations like Eq. (4.6).

After the aperture is closed, the observer must measure the field in the cavity in some way in order to decide whether it contains a component that can be attributed to the signal. The optimum detection operator prescribed in Section 1 cannot easily be found, however, because the operators ρ_0 and ρ_1 do not commute unless the matrices φ_0 and φ_1 commute, and this will not generally be so. We therefore look instead for a threshold detection statistic of the type proposed in Section 3. It will be found by solving Eq. (3.16) under the assumption that the signal is much weaker than the noise in all modes: $(\varphi_s)_{km} \ll (\varphi_0)_{km}$.

The calculation is precisely the same as the one in Section 4, and the detection operator Π_θ has the same quadratic form as in Eq. (4.15). The functionals $R_i(\alpha^*, \beta)$, $i = 0, 1$, corresponding to the density operators ρ_0 and ρ_1 through Eq. (4.9) are given by Eq. (4.12) with $\mathbf{M} = 0$ and φ replaced by φ_0 and φ_1, respectively. When the signal strength is small, the functional $R_1(\alpha^*, \beta)$ is approximately

$$\begin{aligned} R_1(\alpha^*, \beta) &\doteq R_0(\alpha^*, \beta)[1 + \mathbf{A}^+\mathbf{K}\mathbf{B} - \mathrm{Tr}\,(\mathbf{I} + \varphi_0)^{-1}\varphi_s] \\ \mathbf{K} &= (\mathbf{I} + \varphi_0)^{-1}\varphi_s(\mathbf{I} + \varphi_0)^{-1}. \end{aligned} \tag{5.3}$$

The matrix $\mathbf{Q}$ entering the detection operator is then the solution of the equation

$$2\varphi_s = \varphi_0\mathbf{Q}(\mathbf{I} + \varphi_0) + (\mathbf{I} + \varphi_0)\mathbf{Q}\varphi_0 . \tag{5.4}$$

As before the constant b' makes the expected value $\mathrm{Tr}\,(\rho_0\Pi_\theta)$ vanish. It is given by

$$b' = -\mathrm{Tr}\,[(\mathbf{I} + \varphi_0)^{-1}\varphi_s]. \tag{5.5}$$

In the representation in which the matrix φ_0 is diagonal, $\varphi_0 = \| N_k'\delta_{km} \|$, the threshold detection operator takes the form

$$\Pi_0 = b' + \sum_k \sum_m [\tfrac{1}{2}(N_k' + N_m') + N_k'N_m']^{-1} a_k^+(\varphi_s)_{km}a_m , \tag{5.6}$$

again within a constant factor. If the density operators ρ_0 and ρ_1 happen to commute, the matrices φ_0 and φ_s are simultaneously diagonalizable; and if we call their eigenvalues N_k and m_k, respectively, the threshold operator becomes

$$\Pi_0 = b' + \sum_k [N_k(N_k + 1)]^{-1} m_k a_k^+ a_k , \tag{5.7}$$

and has the same form as one given previously (Helstrom, 1965).

As with the threshold operator derived in Section 4, the false-alarm and detection probabilities for a receiver in which this detection operator is measured are difficult to calculate. The moment-generating functions of the operator

$$Q' = \sum_{\mathrm{k,m}} Q_{\mathrm{km}} a_{\mathrm{k}}{}^{+} a_{\mathrm{m}}$$

are now given under the two hypotheses by

$$\begin{aligned} \mathrm{Tr}\,(\rho_i e^{uQ'}) &= |\det(\mathbf{I} - \mathbf{P}\varphi_i)|^{-1} \\ &= \exp\left[-\mathrm{Tr}\ln(\mathbf{I} - \varphi_i \mathbf{P})\right] \qquad i = 0, 1 \qquad (5.8) \\ \mathbf{P} &= \exp(u\mathbf{Q}) - \mathbf{I} \end{aligned}$$

from which the cumulants for substitution in Edgeworth series can be derived.

The equivalent signal-to-noise ratio D^2 of the threshold receiver is, according to Eq. (3.17),

$$\begin{aligned} D^2 &= \mathrm{Tr}\,[(\rho_1 - \rho_0)\Pi_\theta] = \mathrm{Tr}\,(\mathbf{Q}\varphi_1) - \mathrm{Tr}\,(\mathbf{Q}\varphi_0) = \mathrm{Tr}\,(\mathbf{Q}\varphi_s) \\ &= \sum_{\mathrm{k,m}} [\tfrac{1}{2}(N_{\mathrm{k}} + N_{\mathrm{m}}) + N_{\mathrm{k}}N_{\mathrm{m}}]^{-1}(\varphi_s)^2_{\mathrm{km}} \end{aligned} \qquad (5.9)$$

when the representation with φ_0 diagonal is used. If φ_s is then also diagonal,

$$D^2 = \sum_{\mathrm{k}} [N_{\mathrm{k}}(N_{\mathrm{k}} + 1)]^{-1} m_{\mathrm{k}}{}^2 \qquad (5.10)$$

as before (Helstrom, 1965). When a large number M of independent exposures are possible, the false-alarm and detection probabilities are given approximately by Eq. (3.3) in terms of this equivalent signal-to-noise ratio D^2.

6. CONCLUSION

The task of realizing the threshold receivers of Sections 4 and 5 remains. The cavity must be designed in such a way as to maximize the probability of detection attainable by the threshold receiver, and a physical means of combining the mode amplitudes as directed by Eqs. (4.23) and (5.6) must be discovered, but these are problems in electromagnetism. Signal detection theory can only specify the measurements to be made on the receiver and their subsequent treatment for the purpose of decision.

Much more is yet to be done. The structure and properties of the optimum detection operator put forth in Section 1 should be explored in situations where the two density operators ρ_0 and ρ_1 do not commute. Parameter estimation, as formulated in Eq. (1.16), should be studied, and the quantum-mechanical counterparts of minimum-mean-square and maximum-likelihood estimation worked out and evaluated. The optimum forms of special detectors, such as those employing photosensitive surfaces and counters, should be derived and compared with the ideal receiver.[3] The multifarious special problems of optical radar and communication systems, arising from the vagaries of the transmitting medium and the sources of the radiation, provide further opportunities for applying the methods of signal-detection theory and should not be neglected.

APPENDIX A. PROPERTIES OF THE DETECTION OPERATOR

In the representation in which the matrix of the detection operator Π is diagonal, as given in Eq. (1.2), its diagonal elements are the eigenvalues π_k, which lie between 0 and 1, $0 \leqq \pi_k \leqq 1$. Let another representation be obtained by a transformation specified by the unitary matrix $\mathbf{U} = \| U_{km} \|$. The new matrix elements of Π are

$$\Pi_{km} = \sum_{n,p} U_{kn}\pi_n\delta_{np}U_{pm}^{+} = \sum_n \pi_n U_{kn}U_{mn}^{*} . \tag{A.1}$$

In particular the diagonal elements are

$$\Pi_{kk} = \sum_n \pi_n | U_{kn} |^2 \leqq \sum_n | U_{kn} |^2 = 1, \tag{A.2}$$

in which the final step follows from the unitarity of $\mathbf{U}$. The π_n's being all positive, Π_{kk} must be positive, and we have shown that

$$0 \leqq \Pi_{kk} \leqq 1. \tag{A.3}$$

The operator Π^2 has eigenvalues π_m^2, and since $0 \leqq \pi_m^2 \leqq 1$, Π^2 is an operator of the same kind as Π. In any representation its diagonal elements lie between 0 and 1:

$$0 \leqq (\Pi^2)_{mm} = \sum_k \Pi_{mk}\Pi_{km} = \sum_k | \Pi_{km} |^2 \leqq 1, \tag{A.4}$$

where we have used the Hermiticity of Π. This equation shows that all the matrix elements of Π have absolute values less than or equal to 1:

$$| \Pi_{km} | \leqq 1. \tag{A.5}$$

[3] Two recent papers (Helstrom, 1964, 1967) represent a start in this direction.

If we now write Eq. (A.1) as

$$\Pi_{km} = \sum_n \pi_n^{1/2} U_{kn} \cdot \pi_n^{1/2} U_{mn}^*$$

and apply Schwarz's inequality, we find, using Eq. (A.2),

$$| \Pi_{km} |^2 \leqq \sum_n \pi_n | U_{kn} |^2 \cdot \sum_n \pi_n | U_{mn} |^2 = \Pi_{kk} \Pi_{mm} . \qquad \text{(A.6)}$$

Hence any off-diagonal element Π_{km} has an absolute value less than or equal to the geometric mean of the diagonal elements of Π in its row and column. If either of these diagonal elements is 0, the off-diagonal element must also vanish.

APPENDIX B. A GENERATING FUNCTION FOR THE CUMULATIVE LAGUERRE DISTRIBUTION

Dropping subscripts we write the cumulative Laguerre distribution of Eq. (2.20) as

$$\begin{aligned} p_k &= (1 - v) e^{-Nx} v^k L_k(-x), \\ x &= N_s / N(N + 1), \qquad N = v/(1 - v). \end{aligned} \qquad \text{(B.1)}$$

It can be derived from the generating function (Erdélyi *et al.*, 1953, Vol. 2, Eq. (10.12 (18)), p. 189),

$$f(u) = \sum_{k=0}^{\infty} u^k p_k / k! = (1 - v) e^{-Nx} e^{uv} I_0(2\sqrt{xuv}). \qquad \text{(B.2)}$$

Define the generating function of the cumulative distribution

$$q_k = \sum_{j=k}^{\infty} p_j \qquad \text{(B.3)}$$

by

$$g(u) = \sum_{k=0}^{\infty} u^k q_k / k!. \qquad \text{(B.4)}$$

Then differentiation with respect to u shows that it satisfies the differential equation

$$\begin{aligned} g(u) - g'(u) &= f(u), \\ g(0) &= 1. \end{aligned} \qquad \text{(B.5)}$$

The solution of this equation is

$$
\begin{aligned}
g(u) &= e^{u} \int_{u}^{\infty} e^{-y} f(y)\, dy \\
&= (1 - v) e^{-Nx} e^{u} \int_{u}^{\infty} e^{-(1-v)y} I_0(2\sqrt{xvy})\, dy \\
&= e^{u} Q(\sqrt{2Nx}, \sqrt{2(1 - v)u}),
\end{aligned}
\tag{B.6}
$$

where $Q(\alpha, \beta)$ is the Q-function as given at the end of Section 2. In terms of this generating function the cumulative distribution is

$$
q_k = \frac{d^k}{du^k} g(u)|_{u=0} . \tag{B.7}
$$

Received: April 29, 1966.

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